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Minimum-Time Continuous-Thrust Orbit Transfers Using the Kustaanheimo–Stiefel Transformation

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Introduction

IN this Note, we develop the minimum-time, continuous-thrust, planar transfer from one circular orbit to another using the Kustaanheimo–Stiefel (KS) transformation. The development is closely related to other work on minimum-time, continuous-thrust transfers[1] and makes use of the approximate initial values of Lagrange costates developed in Ref. 1. The following development is intended to provide insight into the numerical solutions.

The KS transformation[2] is intended to regularize the equations of motion in the problem of two bodies.[3] When this transformation is used in conjunction with a change of independent variable, the equations of motion in two dimensions have the form of a harmonic oscillator.[2] This allows for simple analytical solutions, which may be perturbed by other forces such as a third body or a propulsion system. If the spacecraft is propelled by continuous thrust, there are no analytical solutions to the problem. However, a perturbation approach using a small thrust parameter may provide a reasonable approximation to the exact case. To determine the accuracy of such an approximation, an exact numerical case must be available. Because the exact solution requires solving a two-point boundary-value problem, it is of interest to develop a reliable means of obtaining these solutions.

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Euler–Lagrange theory is applied to the transformed equations of motion to obtain the optimal control formulation. Based on the results of many numerical cases, the optimal initial costates are plotted against each other for a large range of initial thrust acceleration values. A specific example is provided of an Earth-to-Mars transfer, with the resulting converged values of the initial costates and the fictitious time.

Equations of Motion

We begin by developing the regularized equations of motion, including the terms due to continuous thrust acceleration $a(t) = T/(m_0 + \dot{m}t)$, where T is the constant thrust magnitude, m_0 is the initial mass, and $\dot{m}$ is the constant rate at which mass is expelled by the thruster. All quantities are expressed in canonical units,[4] where the gravitational constant μ is unity regardless of the system under consideration as long as the initial circular radius is defined to be one distance unit (DU) and the initial circular velocity is one distance unit per time unit (DU/TU). The initial acceleration due to thrust is $A = a(t_0)$, and the final desired orbit radius is $R = r(t_f)$. The equations of motion for a two-dimensional orbit are as follows:

$$\ddot{x} = -(\mu/r^3)x \tag{1}$$

$$\ddot{y} = -(\mu/r^3)y \tag{2}$$

as seen in Fig. 1.

Using the KS transformation for two dimensions, the coordinates x and y are replaced by u_1 and u_2 through the following relationship:

$$(u_1 + iu_2)^2 = x + iy \tag{3}$$

The independent variable t is replaced by the fictitious time s with the following differential equation:

$$\frac{dt}{ds} = r \tag{4}$$

These transformations lead to the regularized equations of motion[2]

$$u_1'' = \left[\frac{2(\boldsymbol{u}'^T\boldsymbol{u}') - \mu}{2r}\right]u_1 \tag{5}$$

$$u_2'' = \left[\frac{2(\boldsymbol{u}'^T\boldsymbol{u}') - \mu}{2r}\right]u_2 \tag{6}$$

The primes indicate differentiation with respect to s, $\boldsymbol{u} = (u_1, u_2)^T$, and $r = u_1^2 + u_2^2$. The symbol $(\boldsymbol{u}'^T\boldsymbol{u}')$ indicates an inner product. The purpose of the regularization under the KS transformation is to reduce numerical integration difficulties when r is small by placing the inverse of r into a term that represents the constant angular momentum magnitude of a two-body orbit. This term premultiplies the state variables u_1 and u_2 in Eqs. (5) and (6) but it will not remain

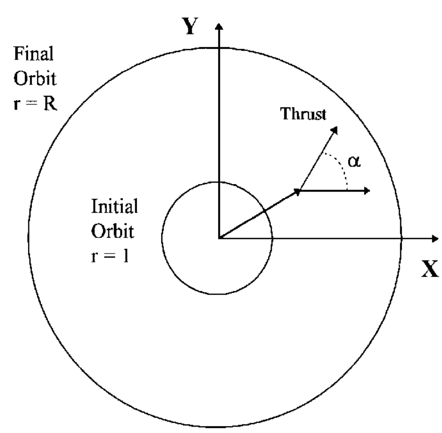

Fig. 1 Problem geometry in two dimensions.

constant with the influence of thrust. A thrust model may be added as follows:

$$u_1'' = \left[\frac{2(\boldsymbol{u}'^T\boldsymbol{u}') - \mu}{2r}\right]u_1 + \frac{1}{2}Ar^{\frac{3}{2}}\cos\gamma \tag{7}$$

$$u_2'' = \left[\frac{2(\boldsymbol{u}'^T\boldsymbol{u}') - \mu}{2r}\right]u_2 + \frac{1}{2}Ar^{\frac{3}{2}}\sin\gamma \tag{8}$$

where A is the magnitude of the thrust acceleration. This is an original thrust model, which is consistent with the transformation given by Stiefel and Scheifele [Ref. 2, Eq. (9.26)], but here the thrust angle γ has been defined in the u coordinate system for simplification with no loss of generality. The relationship between the inertial Cartesian thrust angle α and the KS thrust angle γ is as follows:

$$\cos\gamma = r^{-\frac{1}{2}}(u_1\cos\alpha + u_2\sin\alpha) \tag{9}$$

$$\sin\gamma = r^{-\frac{1}{2}}(u_1\sin\alpha - u_2\cos\alpha) \tag{10}$$

At the initial time, $u_1 = 1$, $u_2 = 0$, and $r = 1$. Therefore, $\gamma(0) = \alpha(0)$. Defining $v_i = u_i'$, the equations of motion and differential constraints may be expressed as five first-order differential equations:

$$t' = r \tag{11}$$

$$u_1' = v_1 \tag{12}$$

$$u_2' = v_2 \tag{13}$$

$$v_1' = \left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]u_1 + \frac{1}{2}Ar^{\frac{3}{2}}\cos\gamma \tag{14}$$

$$v_2' = \left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]u_2 + \frac{1}{2}Ar^{\frac{3}{2}}\sin\gamma \tag{15}$$

Although it is possible to use the KS transformation for problems with three dimensions,[3] only the two-dimensional problem is addressed here. In the minimum-time problem,[5] the cost functional is the real time. The independent variable has been changed to s under the KS transformation. Therefore, the cost functional becomes

$$\mathcal{J} = t_f = \int_{s=0}^{s=s_f} r\,\mathrm{d}s \tag{16}$$

Thus, the Lagrangian is r, and the Hamiltonian for this problem is

$$H = \lambda_{u_1}v_1 + \lambda_{u_2}v_2 + \lambda_{v_1}\left\{\left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]u_1 + \frac{1}{2}Ar^{\frac{3}{2}}\cos\gamma\right\}$$
$$+ \lambda_{v_2}\left\{\left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]u_2 + \frac{1}{2}Ar^{\frac{3}{2}}\sin\gamma\right\} + r + \tilde{\lambda}_t r \tag{17}$$

Defining $\lambda_t = \tilde{\lambda}_t + 1$, the Hamiltonian becomes

$$H = \lambda_{u_1}v_1 + \lambda_{u_2}v_2 + \lambda_{v_1}\left\{\left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]u_1 + \frac{1}{2}Ar^{\frac{3}{2}}\cos\gamma\right\}$$
$$+ \lambda_{v_2}\left\{\left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]u_2 + \frac{1}{2}Ar^{\frac{3}{2}}\sin\gamma\right\} + \lambda_t r \tag{18}$$

The optimal control law is found by setting $\partial H/\partial\gamma$ equal to zero. For a minimum, the result is

$$\tan\gamma = -\lambda_{v_2}/-\lambda_{v_1} \tag{19}$$

This choice of sign satisfies the Legendre–Clebsch condition for a minimum. From this control law, we have the following:

$$\cos\gamma = \frac{-\lambda_{v_1}}{\sqrt{\lambda_{v_1}^2 + \lambda_{v_2}^2}} \tag{20}$$

$$\sin\gamma = \frac{-\lambda_{v_2}}{\sqrt{\lambda_{v_1}^2 + \lambda_{v_2}^2}} \tag{21}$$

Table 1 Initialization for KS problem

$u_1(0) = 1$	$\lambda_{u_1}(0) = 1$
$u_2(0) = 0$	$\lambda_{u_2}(0) = ?$
$v_1(0) = 0$	$\lambda_{v_1}(0) = 2\lambda_{u_2}(0)$
$v_2(0) = \frac{1}{2}$	$\lambda_{v_2}(0) = ?$

These relationships may be used to eliminate the $\sin\gamma$ and $\cos\gamma$ terms from the Hamiltonian. The costate equations are then found using the canonical relationship $\boldsymbol{\lambda}' = -\partial H/\partial\boldsymbol{q}$, in which $\boldsymbol{q} = (u_1, u_2, v_1, v_2, t)$. Recall that the primes indicate differentiation with respect to the fictional time s. Using this relationship and taking the indicated partial derivatives produces the following five first-order differential equations:

$$\lambda_t' = \frac{-\dot{m}Tr^{\frac{3}{2}}}{2(m_0 + \dot{m}t)^2}\sqrt{\lambda_{v_1}^2 + \lambda_{v_2}^2} \tag{22}$$

$$\lambda_{u_1}' = \left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]\left[\frac{2u_1}{r}(\lambda_{v_1}u_1 + \lambda_{v_2}u_2) - \lambda_{v_1}\right]$$
$$+ \left(\frac{3}{2}\right)Ar^{\frac{1}{2}}u_1\sqrt{\lambda_{v_1}^2 + \lambda_{v_2}^2} - 2u_1 \tag{23}$$

$$\lambda_{u_2}' = \left[\frac{2(\boldsymbol{v}^T\boldsymbol{v}) - \mu}{2r}\right]\left[\frac{2u_2}{r}(\lambda_{v_1}u_1 + \lambda_{v_2}u_2) - \lambda_{v_2}\right]$$
$$+ \left(\frac{3}{2}\right)Ar^{\frac{1}{2}}u_2\sqrt{\lambda_{v_1}^2 + \lambda_{v_2}^2} - 2u_2 \tag{24}$$

$$\lambda_{v_1}' = (-2v_1/r)(\lambda_{v_1}u_1 + \lambda_{v_2}u_2) - \lambda_{u_1} \tag{25}$$

$$\lambda_{v_2}' = (-2v_2/r)(\lambda_{v_1}u_1 + \lambda_{v_2}u_2) - \lambda_{u_2} \tag{26}$$

If $\dot{m} = 0$, then $\lambda_t' = 0$ as well. In this case, λ_t will be a constant. Assuming $\lambda_t \neq 0$, it is possible to divide the Hamiltonian by λ_t, which would scale the remaining costate variables, and eliminate λ_t from the problem. If $\dot{m} \neq 0$, λ_t must be retained because the real time appears explicitly in the equations of motion. The Hamiltonian may be set equal to zero at the initial time by adding an arbitrary constant because the constant contributes nothing to the partial derivatives. Then, it is possible to solve for initial λ_t (or λ_{v_2}) as a function of the initial values of the remaining states and costates:

$$\lambda_t(0) = -\tfrac{1}{2}A_0\sqrt{\lambda_{v_1}^2(0) + \lambda_{v_2}^2(0)} \tag{27}$$

At the initial time, $\lambda_{v_1} = 2\lambda_{u_2}$. This may be shown by equating the KS Hamiltonian to another Hamiltonian in polar coordinates, but with fictitious time. Also, the Hamiltonian may be scaled such that $\lambda_{u_1} = 1$. Thus, there are three remaining values that must be found to solve the boundary-value problem of coplanar transfer between two circular orbits. As shown in Table 1, they are s_f, $\lambda_{u_2}(0)$, and $\lambda_{v_2}(0)$. If λ_t is not used, then the three values that must be found are s_f, $\lambda_{u_1}(0)$, and $\lambda_{u_2}(0)$. The other two costates are found from $\lambda_{v_1} = 2\lambda_{u_2}$ and from solving $H(0) = 0$ for λ_{v_2}. Because the final circular orbit may be described using three scalar values, the number of unknowns is the same as the number of end conditions to be matched.

Initial Costate Locus

Suppose $R = 2$, and we have a converged case for $A = 100$. If A is multiplied by 0.9, then convergence may be achieved using the last values of the initial costates as the new starting guess. If this procedure is repeated, the shooting method will take more iterations to converge and eventually will not converge at all. Then, the multiplication factor must be increased to, perhaps, 0.95 and higher still as A decreases. This phenomenon is due to the sensitivity of the system to initial conditions, which increases for the increased flight times associated with small thrust values at a given R. Although

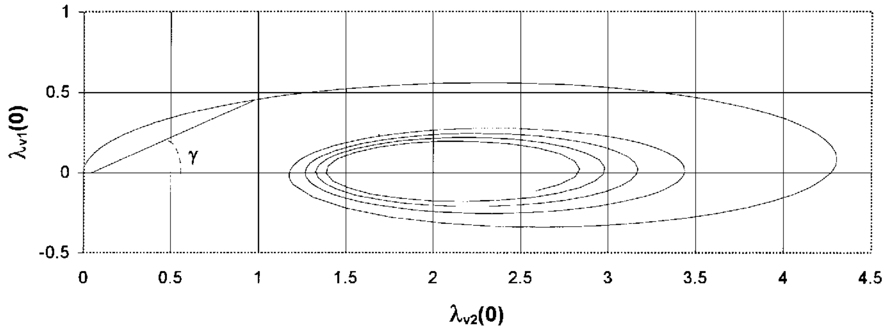

Fig. 2 Optimal initial costate locus under the KS transformation: ——, R = 2.

Table 2 Bryson and Ho[5] example under KS transformation

s_f	$\lambda_{u2}(0)$	$\lambda_{v2}(0)$
2.7090520	0.5371110	2.3409681

this process is difficult and time consuming, it does produce valuable information. After completing the process for a large range of A, it is instructive to make a plot of the converged initial values of the costates vs one another to examine their behavior.[1,6] This choice of axes is motivated by the definition of the thrust angle, $\tan\gamma = -\lambda_{v_2}/-\lambda_{v_1}$. As shown in Fig. 2, the initial value of γ may be measured from the $\lambda_{v_2}(0)$ axis to a point on the locus with the vertex at the origin. Thus, the initial value of γ may be seen directly from Fig. 2. For large values of A, γ approaches 90 deg, and for small values of A, γ approaches 0 deg.

Numerical Example

Next, the familiar Bryson and Ho example[5] of an Earth-to-Mars transfer is presented with $R = 1.525$, $A = 0.1405$, and $\dot{m} = -0.07488$. Table 2 shows the optimal initial costate values under the KS transformation. If the initial costate values shown in Table 2 are used with Table 1 to initialize the differential equations for the states and costates, the desired end conditions will be met at the given value of s_f.

Conversely, the real-time problem may be solved first to obtain the exact value of the fictional time. Once the problem is solved using the real time, the optimal value of s may be obtained through numerical integration of $ds/dt = 1/r$, and the initial KS costate values may be approximated[7] by $\lambda_{v_1}(0) = \lambda_u(0)$ and $\lambda_{v_2}(0) = 2\lambda_v(0)$, where u and v refer to velocity components in polar coordinates.[5] Clearly, having the exact value of the optimal final value of s makes the search for the optimal initial values of $\lambda_{v_1}(0)$ and $\lambda_{v_2}(0)$ much easier. In this way, many example cases may be generated for further study, such as for comparisons with analytical approximations.

Conclusions

The minimum-time continuous-thrust transfer from one coplanar circular orbit to another is examined using the KS transformation. Euler–Lagrange theory is applied to the transformed equations of motion. The optimal initial costates are plotted against each other for a large range of initial thrust acceleration values. A specific example is provided of an Earth-to-Mars transfer with the converged values of the initial costates and the fictitious time. The initial costate locus may be used to aid in finding a numerical example to verify analytical techniques using the KS transformation.

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